# A Transition in Gap Probabilities - from 1 Gap to 2

- Let $K_s$ be the operator on $L^2(A)$ with the kernel

$$K_s(x,y) = \frac{\sin s(x-y)}{\pi(x-y)}.$$

- For a wide class of random matrices , the probability of finding no eigenvalues of a random matrix $M$ on a set $sA/\pi$ in the bulk scaling limit is given by the Fredholm determinant $\det(I - K_s)_A$.
- We focus on the asymptotic behaviour of $\det(I - K_s)_A$ as $s \to \infty$, in particular the probability of large gaps $sA/\pi$ where $A$ is composed of one or two intervals.

- We have seen results for 1 interval and 2 intervals.
- The formula for 2 intervals held on

$$A = ( \quad ; \quad ) \; [ \; ( \quad ; \quad );$$

  for fixed $\; ; \; ; \;$ as $s \to \infty$.
- We study the case where $\; \to 0$ simultaneously as $s \to \infty$. We present results for the regime $s \quad \to 0$.

## Theorem (F, Krasovsky)

For the constant $\quad = \frac{4}{1 \quad\quad 1}$, write $\quad$ in the form

$$= \quad e^{\frac{s\sqrt{j \quad j}}{w}}; \quad w = k + x; \quad k \in \mathbb{N}; \ x \in [\ 1/2; 1/2).$$

Then, as $s \to \infty$ uniformly for $\quad \in (0; \ _0)$, where $s \ _0 \to 0$,

$$\log\det(I \quad K_s)_A = \log\det(I \quad K_s)_{( \ ; \ )}$$
$$+ s\sqrt{j \quad j} \quad w \quad \frac{x^2}{w} \quad + c(k) + \quad (x) + O(\max\{s \ _0; s^{\ 1}\});$$

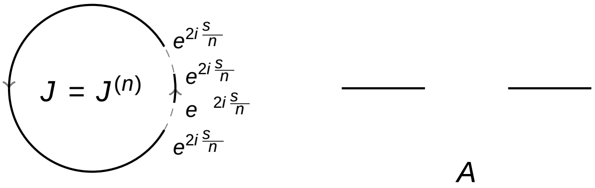
Let $G$ be the Barnes' $G$-function ($G(k+1) = \ (k)G(k)$ and $G(1) = 1$). Then

$$c(k) = \log \left( \frac{2^{2k^2 \quad k}}{k} \frac{G(k+1)^4}{G(2k+1)} \right) :$$

- Alternatively, let $j$

- This transition from one gap to two gaps has an interesting parallel in the unitary random matrix ensembles exhibiting a transition from one-cut support to two-cut support as in the "birth of a cut", where asymptotic results for the correlation kernel where given simultaneously by Bertola & Lee, Claeys, Mo ('07)–('09). Fluctuations of the same type were also witnessed here.

# Method of proof




- Define the Toeplitz determinant on the interval $J$ by

$$D_n(\;) = \det(f_{j\quad k})_{j;k=1}^{n};\qquad f_j = \mathop{Z}_{J} e^{\ \ ij}\ \frac{d}{2}$$

- We have the following link between the Toeplitz determinant and the Fredholm determinant

$$\lim_{n\,!\,1} D_{n;s}(\;) = \det(I \quad K_s)_A:$$

- We apply the Deift-Zhou steepest descent method to obtain asymptotics for $Y$ as $n \to \infty$, in the double scaling limit $s \to \infty$, $\to 0$, $s/n \to 0$ and $s \to 0$, evaluated at the points $e^{2i\frac{s}{n}}$.
- Thus we obtain asymptotics for $\frac{\partial}{\partial} \log D_{n;s}(\ )$ and obtain asymptotics in the same double-scaling regime.
- Thereafter we integrate

$$\int_0^{0} \frac{\partial}{\partial} \log D_{n;s}(\ ) d \ ;$$

and use the identity between Toeplitz determinants and Fredholm determinants to obtain our results.