

Truncations of random unitary matrices revisited

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joint review with H.-J. Sommers:

Non-Hermitian Random Matrix Ensembles, arXiv:0911.5645,
to appear in the Oxford Handbook of Random Matrix Theory

Setup

Choose a unitary matrix at random and partition it:

$$U = \begin{pmatrix} \mathbf{T} & \mathbf{S} \\ \mathbf{Q} & \mathbf{R} \end{pmatrix} \rightarrow \mathbf{T} \quad \mathbf{T} \text{ is } m \times$$

What truncations are good for?

- (1) **Quantum transport problems** (Beenakker'97, poster by Nick Simm)

Additive stats of EVs of $\mathbf{T}\mathbf{T}^\dagger$ describe phys quantities of interest, i.e. $\text{tr}\mathbf{T}\mathbf{T}^\dagger$ for conductance of quasi one-dimensional wires

- (2) **Open chaotic sys** (Fyodorov & Sommers, '97 Życzkowski & S. '00)

Eigenvalues of \mathbf{T} are used to model resonances

- (3) **Combinatorics of vicious walkers**(Novak '09)

$\langle |\text{tr}\mathbf{T}|^N \rangle_T$ enumerates configs of random-turn vicious walkers

- (4) **Random determinants** (Fyodorov & K., '07), e.g.,

$$\left\langle \frac{1}{|\det(\mathbf{I} - \mathbf{z}\mathbf{A})|^2} \right\rangle_A = \int \left\langle \frac{1}{\det(\mathbf{I} - |\mathbf{z}|^2\mathbf{T}\mathbf{T}^\dagger - \mathbf{A}\mathbf{A}^\dagger)} \right\rangle_A d_n \times (\mathbf{T})$$

for complex random $\mathbf{n} \times \mathbf{n}$ matrices \mathbf{A} with invariant distribution.

Singular values of \mathbf{T} (1,4); eigenvalues of \mathbf{T} (2,3)

Matrix measure

Truncation map: $U \mapsto T$, U is $n \times n$, T is $m \times p$, $m \leq p$

Have $TT^\dagger + SS^\dagger = I$ by unitarity. If $n \geq m + p$ then (generically) SS^\dagger has rank m and the image of $U(n)$ is the entire matrix ball $TT^\dagger \leq I$.

Theorem 1 (Friedman&Mello '85, Fyodorov&Sommers '03, Forrester '06)

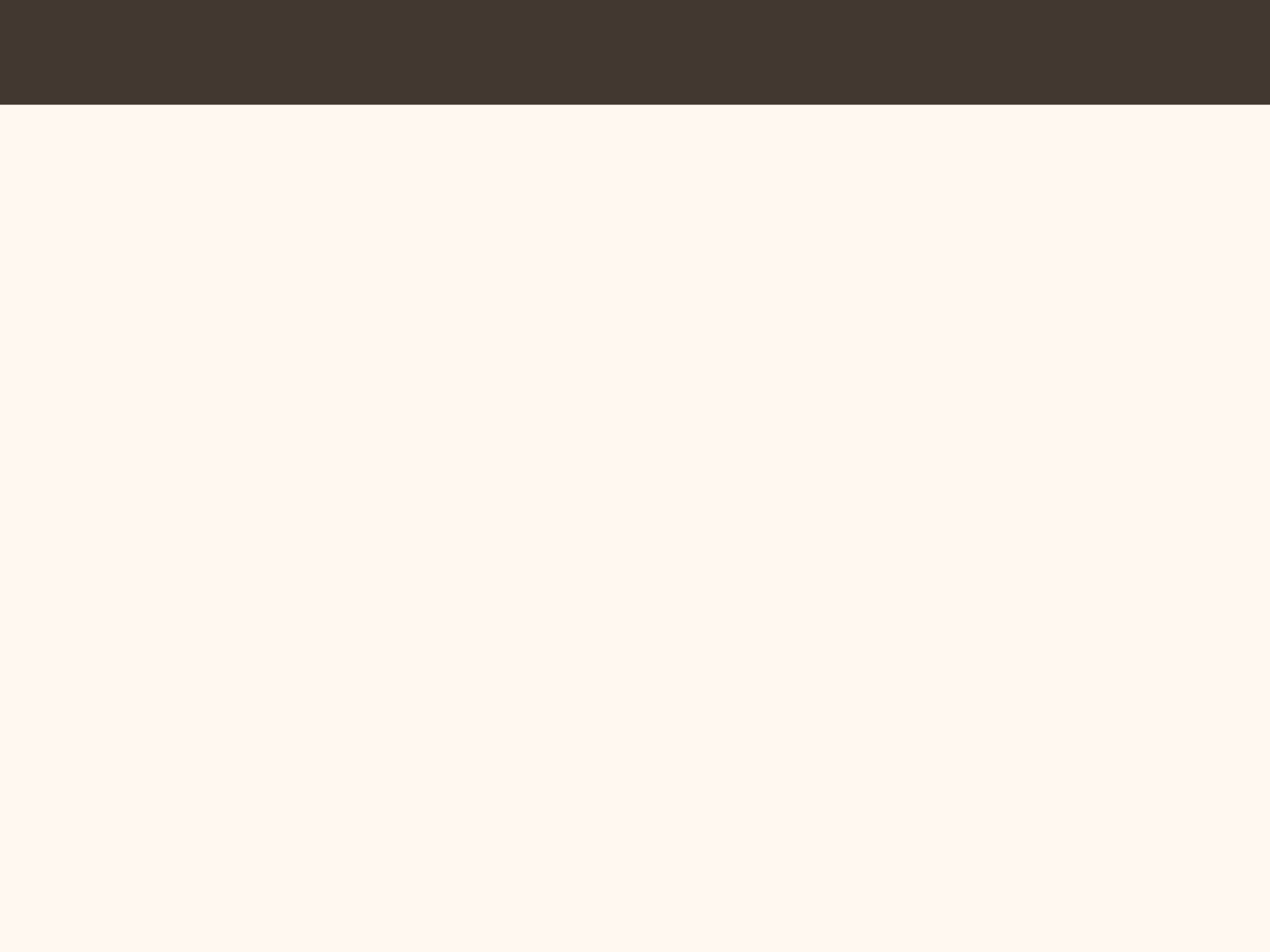
For $n \geq m + p$

$$d_{n \times p}(\mathbf{T}) \det(\mathbf{I} - \mathbf{T}\mathbf{T}^\dagger)^{n-p} \leq (\mathbf{T})d\mathbf{T}$$

where $d\mathbf{T}$ is the Cartesian volume element in $\mathbb{C}^{m \times p}$. For invariant f

$$\begin{aligned} \int_{\mathbb{C}^{m \times p}} f(\mathbf{T}\mathbf{T}^\dagger) d_{n \times p}(\mathbf{T}) &= \text{const.} \times \\ &= \int_{\mathbb{C}^{m \times m}} f(\mathbf{Z}\mathbf{Z}^\dagger) \det(\mathbf{Z}\mathbf{Z}^\dagger)^{p-m} \det(\mathbf{I} - \mathbf{Z}\mathbf{Z}^\dagger)^{n-p} \leq (\mathbf{Z})d\mathbf{Z} \end{aligned}$$

Matrix measure



Eigenvalue correlation functions

These are just marginals of the jpdf:

$$R(z_1, \dots, z_k) = \frac{m!}{(m-k)!} \int d^2z_{k+1} \dots \int d^2z_m P(z_1, \dots, z_k, z_{k+1}, \dots, z_m),$$

The EV corr fncs for truncations can be obtained by the method of OPs.

For the rotation invariant weights, $w(z) = w(|z|)$, OPs are just powers z^l : $\int d^2z w(z) z^l z^{*l} = h_l$, leading to

$$R(z_1, \dots, z_k) = \prod_{l=1}^k w(z_l) \det(\mathbf{K}(z_i, z_j)); \quad \mathbf{K}(u, v) = \sum_{l=0}^{m-1} \frac{(uv^*)^l}{h_l}$$

For truncated unitaries the sum on the rhs is the **binomial series** for $(1 - uv^*)^{-(n-k+1)}$ truncated after m terms. This gives the kernel in terms of the incomplete Beta function.

Incomplete Beta fnc:

$$I_x(\mathbf{a}, \mathbf{b}) = \frac{1}{\mathbf{B}(\mathbf{a}, \mathbf{b})} \int_0^x t^{\mathbf{a}-1} (1-t)^{\mathbf{b}-1} dt$$

We have

$$\mathbf{K}(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{n} - \mathbf{m}}{\mathbf{B}(\mathbf{n} - \mathbf{m} + 1, \mathbf{m})} \frac{I_{1 - \mathbf{u}\mathbf{v}^*}(\mathbf{n} - \mathbf{m} + 1, \mathbf{m})}{(1 - \mathbf{u}\mathbf{v}^*)^{\mathbf{n} - \mathbf{m} + 1}}$$

This representation seems to be new. It is convenient for asymptotic analysis. Also one can handle more general $w(\mathbf{z}) = |\mathbf{z}|^{2\mathbf{a}}(1 - |\mathbf{z}|^2)^{\mathbf{b}}$.

Compare with the complex Ginibre $(d\mu(\mathbf{J}) = e^{-\text{tr} \mathbf{J}} d\mathbf{J})$. There $w(\mathbf{z}) = e^{-|\mathbf{z}|^2}$, have truncated exponential series for the kernel:

$$\mathbf{K}(\mathbf{u}, \mathbf{v}) = \frac{\mathbf{n}}{\mathbf{n}!} e^{-\mathbf{u}\mathbf{v}^*} \frac{\Gamma(\mathbf{n}, \mathbf{u}\mathbf{v}^*)}{\Gamma(\mathbf{n})} \quad \text{with} \quad \Gamma(\mathbf{n}, \mathbf{x}) = \int_x^\infty e^{-t} t^{\mathbf{n}-1} dt$$

Strong non-unitarity - EV density in the bulk

Strong non-unitarity: boundary of EV distribution

Strong non-unitarity, locally at the origin

Scale z

Weak non-unitarity: EV density and correlations

Scaling z accordingly, $z = \left(1 - \frac{y}{m}\right) e^{i\varphi_0 + i\frac{\varphi_j}{m}}$, one finds the EV density

$$R_1(z) = \frac{m^2 (2y)^{l-1}}{(l-1)!} \int_0^1 e^{-2t} t^{l-1} dt, \quad m \rightarrow \infty \quad \text{and } l \text{ is finite.}$$

(Życzkowski & Sommers, '00) and correlations

$$R(z_1, \dots, z_l) = \left(\frac{m^2}{2y}\right)^l \prod_{i=1}^l \frac{(2y)^{l-1}}{(l-1)!} \det \left(\int_0^1 e^{-2t} t^{l-1} e^{i(\varphi_i - \varphi_j)} dt \right)$$

This is a particular case of a 'universal' expression describing EV correlations for random contractions (Fyodorov & Sommers '03).

Interestingly, a different ensemble, $J = H + iW$, leads to the same form of correlations (Fyodorov & K. '99). Here H is drawn from the GUE, $\beta > 0$ and W is a diagonal matrix with $l-1$'s and m zeros.

Conclusion