Critical asymptotics for Toeplitz determinants

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Toeplitz determinants

- Toeplitz matrix = matrix which is constant along diagonals

$$\begin{pmatrix} c_0 & c_{\ 1} & c_{\ 2} & \cdots & c_{\ n+1} \\ c_1 & c_0 & c_{\ 1} & \ddots & \vdots \\ c_2 & c_1 & \ddots & \ddots & c_{\ 2} \\ \vdots & \ddots & \ddots & c_0 & c_{\ 1} \\ c_{n\ 1} & \cdots & c_2 & c_1 & c_0 \end{pmatrix}$$

- Toeplitz determinant is the determinant of a Toeplitz matrix
- Asymptotics for Toeplitz determinants when the size of the matrices tends to infinity?

Toeplitz determinants

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Toeplitz determinants

- If the weight $f \;\; \theta$
  - is "smooth"
  - has no zeros
  - has a continuous logarithm (winding number around the origin)
- Szegő's strong limit theorem: as n ,

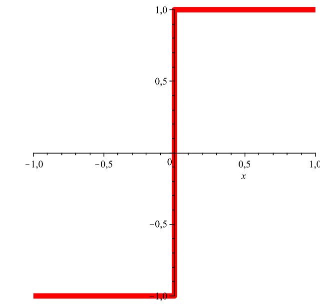
$$\ln D_n \;\; f \qquad n \ln f \;\; {}_0 + \qquad \underset{k=1}{} \; k \;\ln f \;\; {}_k \; \ln f \;\; {}_k + o \qquad ,$$

with

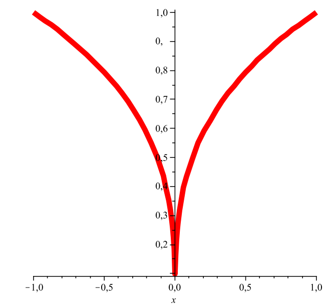
$$\ln f \;\; {}_k \qquad \frac{}{\pi} \quad {}_0^{2\pi} \; \ln f \;\; e^{i\theta} \;\; e^{\;\;ik\theta} d\theta.$$

Fisher-Hartwig singularities

- Two types of weights for which Szegő asymptotics are not valid
  - jump discontinuities



  - root type singularities



- Example

$$f \quad e^{i\theta} \qquad \cos\theta \ {}^{\alpha} e^{i\beta(\theta \quad \pi)} e^{V(e^{i\theta})}, \qquad \text{for} \quad < \theta < \quad \pi,$$

with $\operatorname{Re}\alpha > \quad 1$

Fisher-Hartwig singularities

- For weights with one Fisher-Hartwig singularity with parameters α (root) and β (jump),

$$\ln D_n \; f \qquad nV_0 + \sum_{k=1}^{\infty} kV_kV_{-k} \qquad \alpha \quad \beta \sum_{k=1}^{\infty} V_k \qquad \alpha+\beta \sum_{k=1}^{\infty} V_{-k}$$

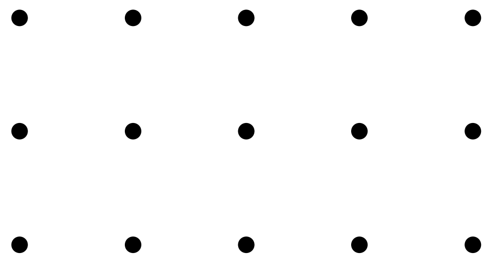
$$+ \;\; \alpha^2 \quad \beta^2 \; \ln n + \ln \frac{G \quad + \alpha + \beta \; G \quad + \alpha \quad \beta}{G \quad + \;\; \alpha} + o \qquad ,$$

as n , where G is Barnes' G-function, and

$$V_k \qquad —$$

2d Ising model

- lattice with an associated spin variable taking values $\pm$ at each point of the lattice



2d Ising model

- 2-spin correlation functions are Toeplitz determinants:

$$< \sigma_{00}\sigma_{0\mathbf{k}} > \quad D_{\mathbf{k}} \; f \; ,$$

for a certain symbol f

  - For $T < T_c$

Transition from Szegő to FH

Transition from Szegő to FH

- Asymptotics as n

Transition from Szegő to FH

$$\mathrm{nt} \quad {}_{0}^{2\mathbf{nt}} \left(\mathrm{w} \ \mathrm{x} \quad \frac{\alpha^2 \quad \beta^2}{\mathrm{x}} \right) \mathrm{dx} + \ \alpha^2 \quad \beta^2 \ \ln \ \mathrm{nt},$$

$$\mathrm{w} \ \mathrm{x} \quad \frac{}{\mathrm{x}} \ {}_{\mathbf{x}}^{+\infty} \ \mathrm{v} \ \xi \ \mathrm{d}\xi,$$

- V

Asymptotics

$$v\ x \quad \begin{cases} \mathcal{O} \quad + \mathcal{O}\ x^{2\alpha} \ + \mathcal{O}\ x^{2\alpha} \ln x\ , & x \quad , \\ \mathcal{O}\ e^{-cx}\ , & x \quad + \quad , \end{cases}$$

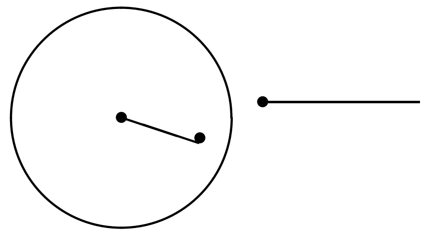
$$\overset{+}{\underset{0}{}} \quad v\ x\ dx \quad \alpha^2 \quad \beta^2.$$

$$w\ x \quad \begin{cases} \frac{\alpha^2-\beta^2}{x} + \mathcal{O} \quad + \mathcal{O}\ x^{2\alpha} \ + \mathcal{O}\ x^{2\alpha} \ln x\ , & x \quad , \\ \mathcal{O}\ e^{-cx}\ , & x \quad + \quad . \end{cases}$$

$$x \quad \begin{cases} \alpha^2 \quad \beta^2\ \ln x + \mathcal{O}\ x\ , & x \quad , \\ \ln \frac{G(1+\alpha+\beta)G(1+\alpha-\beta)}{G(1+2\alpha)} + \mathcal{O}\ e^{-cx}\ , & x \quad + \quad . \end{cases}$$

Transition from Szegő to FH

- Extension to complex t?



Expansion is valid for $\arg t < \frac{\pi}{2}$ if contour of integration does not contain poles of w

- different choices of contour different branches of logarithm

Transition from Szegő to FH

- what if Im α and/or Re β ?
  - ► w x α, β is not real for x >
  - ► w can have poles on , +
  - ► asymptotic expansion holds only if we integrate over a pole-free contour

    expansion not valid if nt is a pole of w x α, β
  - ► poles correspond to Toeplitz determinants approaching

    different choices of integration contour

    expansion picks up residue of w

    residue of w 4.797(p)-0.5a3.959z246]TJ /R28 20

Orthogonal polynomials

- Heine's formula: determinant formula for orthogonal polynomials

p_{nab}

Asymptotics for Toeplitz determinants

General approach to obtain asymptotics for Toeplitz determinants for weight f

- Step 1: deform weight f smoothly to a weight for which Toeplitz determinant is known (e.g. uniform weight),

$$f_t\ z\ , \qquad f_1\ z \quad f, \qquad f_0\ z$$

- Step 2: try to find differential identity for $\frac{d}{dt}$

Transition from Szegő to FH

Applied to our transition between Szegő and FH

- Step 1: deformation of weight:

$$f_t \;\; z \qquad z \qquad e^{t \;\; \alpha+\beta} \;\; z \qquad e^{\;\; t \;\; \alpha}$$

Transition from Szegő to FH

- Step 2: differential identity

$$\frac{d}{dt}\ln D_n\ t \qquad \alpha+\beta\ e^{t}\left(Y^{-1}Y'\right)_{22}\ e^{t} + \alpha\ \ \beta\ e^{-t}\left(Y^{-1}Y'\right)_{22}\ e^{-t}$$

where

$$Y\ \ z \qquad \begin{pmatrix} \chi_n^{-1} p_n\ \ z & p_n^{-1}\ \ C_1\ \frac{p_n(\xi)}{\xi - z}\frac{f(\xi)d\xi}{2\pi i\xi^n} \\ \chi_{n-1} z^{n-1} p_{n-1}\ \ z^{-1} & \chi_{n-1}\ \ C_1\ \frac{p_{n-1}(\xi^{-1})}{\xi - z}\frac{f(\xi)d\xi}{2\pi i\xi} \end{pmatrix}$$

  - Y is solution of the Riemann-Hilbert problem for orthogonal polynomials