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Problem Statement

Let K_s be the integral operator acting on $L^2(-s,s)$ with confluent hypergeometric kernel:

$$K_s(u,v) = \frac{1}{2\pi}\frac{\Gamma(1-\beta+\alpha)\Gamma(1+\beta+\alpha)}{\Gamma(1+2\alpha)\Gamma(2+2\alpha)}\frac{A(u)B(v)-A(v)B(u)}{u-v},$$

$$A(x) = 2e^{-\pi i\beta sgn(x)/2} x|2x|^{\alpha} e^{-ix}\varphi \ldots$$ [illegible]

Theorem (1)

The asymptotics for the Fredholm determinant $P_s = \det(I - K_s)$ *on* $(-s, s)$ *as* $s \to \infty$ *are given by the formula*

$$\ln P_s = -\,s^2$$

Two usual types of endpoints

- the density of eigenvalues vanishes as a square root ("soft edge" of the spectrum, e.g., GUE endpoints of semicircle). In the scaling limit at the endpoint one obtains the Airy kernel:

$$K_{Airy}(x,y) = \frac{\mathrm{Ai}(x)\mathrm{Ai}'(y) - \mathrm{Ai}(y)\mathrm{Ai}'(x)}{x-y}, \qquad \text{on } (-s,\infty)$$

Asymptotics of the Tracy-Widom distribution:

$$\ln\det(I - K_{Airy}) = -\frac{s^3}{12} - \frac{1}{8}\ln s + \kappa + O(s^{-3/2}), \qquad s \to +\infty,$$

$$\kappa = \frac{1}{24}\ln 2 + \zeta'(-1),$$

(Tracy, Widom (1994), Deift, Its, Krasovsky (2008), Baik, Buckingham, DiFranco (2008))

- the density of eigenvalues diverges as a square root ("hard edge" of the spectrum, e.g. the Laguerre ensemble at 0 or Jacobi ensemble at the edgepoints). In the scaling limit at the endpoint one obtains the Bessel kernel:

$$K_{Bes}(x,y) = \frac{\sqrt{x}J_{a+1}(\sqrt{x})J_a(\sqrt{y}) - \sqrt{y}J_a(\sqrt{x})J_{a+1}(\sqrt{y})}{2(x-y)}, \quad (a > -1),$$

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Theorem (2)

The large s asymptotics of $P_s = \det(I - K_{Bes})$ are given by

$$\ln\det(I - K_{Bes}) = -\frac{s}{4} + a\sqrt{s} - \frac{a^2}{4}\ln s + c_1 + O\left(\frac{1}{\sqrt{s}}\right),$$

where

$$c_1 = \ln\frac{G(1+a)}{(2\pi)^{a/2}}.$$

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The main idea is to use a double-scaling limit of a Toeplitz determinant to obtain asymptotics of the Fredholm determinant P_s. The Toeplitz determinant with symbol f is given by the expression:

$$D_n(f) = \det\left(\frac{1}{2\pi}\int_0^{2\pi} e^{-i(j-k)\theta} f(\theta)d\theta\right)_{j,k=0}^{n-1}.$$

Let

$$f_\phi(\theta) = \begin{cases} f(z) = |z-1|^{2\alpha} z^\beta e^{-i\pi\beta}, z = e^{i\theta}, & \phi < \theta < 2\pi - \phi, \\ 0, & otherwise, \end{cases}$$

A classical representation of a Toeplitz determinant:

$$D_n(f_\phi) = \prod_{k=0}^{n-1} \kappa_k^{-2},$$

Here $p_k(z) = \kappa_k z^k + \dots$ are polynomials, orthonormal on the unit circle w.r.t. $f_\phi(\theta)$.

$$\frac{d}{d\phi} \ln D_n(f_\phi) = \text{ in terms of } p_{n-1}(e^{\pm i\phi}), p_n(e^{\pm i\phi}).$$

We find the asymptotics of $p_n(z)$ by solving the associated Riemann-Hilbert problem (RHP).

$$\frac{d}{d\phi} \ln D_n(f_\phi) = -\frac{1}{2} n^2 \tan\frac{\phi}{2} - \frac{n}{2}\left(2\alpha \tan\frac{\phi}{2} - \frac{2\alpha}{\cos\frac{\phi}{2}}\right) - \frac{1}{8}\frac{\cos\phi/2}{\sin\phi/2}$$
$$+ \frac{\beta^2 \cos\phi/2}{2\sin\phi/2} - \alpha^2\left(\tan\frac{\phi}{2} - \frac{1}{\cos\phi/2} + \frac{\cos\phi/2}{2\sin\phi/2}\right) + O\left(\frac{1}{n\sin^2(\phi/2)}\right),$$

where the remainder term is uniform for $2s/n \leq \phi \leq \pi - \varepsilon, \quad \varepsilon > 0$.

sketch of the proof for the Th.1

$D_n(f_\phi)-?$ as $\phi \to \pi$ from below.

$$D_n(f_\phi) = \frac{1}{(2\pi)^n n!} \int_{\phi}^{2\pi-\phi} \cdots \int_{\phi}^{2\pi-\phi} \prod_{1 \le j < k \le n} |e^{i\theta_j} - e^{i\theta_k}|^2 \prod_{j=1}^{n} f(e^{i\theta_j}) d\theta_j .$$

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After a change of variables we obtain:

$$D_n(f_\phi) = \frac{\varepsilon^{n^2} 2^{2\alpha n}}{(2\pi)^n} \left(A_n + O(\varepsilon^2) \right), \quad \phi = \pi - \varepsilon, \quad \varepsilon \to 0,$$

where

$$A_n = \frac{1}{n!} \int_{-1}^{1} \cdots \int_{-1}^{1} \prod_{1 \le i < j \le n} (z_i - z_j)^2 \prod_{j=1}^{n} dz_j \quad - \text{ Selberg integral}$$

The asymptotics of A_n as $n \to \infty$ are known (Widom). Then, for $\phi = \pi - \varepsilon$

$$\ln D_n(f_\phi) = n^2(\ln \varepsilon - \ln 2) + 2\alpha n \ln 2 - \frac{1}{4} \ln n + \frac{1}{12} \ln 2 + 3\zeta'(-1) + O_n(\varepsilon),$$

where $O_n(\varepsilon) \to 0$, as $\varepsilon \to 0$.

Bessel kernel can be obtained as a scaling limit at the endpoint for the polynomials orthogonal on the interval $[-1, 1]$ that are related to the polynomials orthogonal on the unit circle with Fisher-Hartwig weight for $\beta = 0$. Consider the Hankel determinant with symbol $\omega(x)$:

$$D_n^H(\omega) = \det\left(\int_{-1}^{1} x^{j+k}\omega(x)dx\right)_{j,k=0}^{n-1}.$$

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The Fredholm determinant $P_s^{Bes} = \det(I - K_s^{Bes})$:

$$P_s^{Bes} = \lim_{n\to\infty} \frac{D_n^H(\omega_{\frac{2s}{n}})}{D_n^H(\omega)}, \qquad \omega_\phi(x) = \frac{f_\phi(e^{i\theta})}{|\sin(\theta)|}, \quad x = \cos\theta.$$

Connection formula between Toeplitz and Hankel determinants:

$$\left(D_n^H(\omega)\right)^2 = \frac{\pi^{2n}}{2^{2(n-1)^2}} \frac{(\kappa_{2n} + p_{2n}(0))^2}{p_{2n}(1)p_{2n}(-1)} D_{2n}(f(z)), \qquad \text{(Deift, Its, Krasovsky)}$$

here p_n are polynomials orthonormal on the unit circle with the weight $f(z)$.

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