Universality in complex Wishart ensembles

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Eigenvalue statistics of complex Wishart ensembles

Multiple Laguerre polynomials and Riemann-Hilbert problem

Results

$_N$ has distinct eigenvalues, N_0 of them and N of them a.

M N and as $N \to$, both N_0, $N \to$ and $\frac{N}{M} \to c$, $\frac{N}{N} \to$.

Universality: Eigenvalue correlations given by the Sine-kernel in the bulk of the spectrum and Airy kernel in the lim

Global eigenvalue statistics

Studied by Bai, Choi and Silverstein with Stieltjes transform.

$$m_G(z) = \int_{-\infty}^{\infty} \frac{1}{x - z} dG(x), \quad z \in \mathbb{C}^+ \equiv \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$$

The Stieltjes transform of the limiting eigenvalue distribution satisfies

$$m(z) = \int_{\mathbb{R}} \frac{1}{t(1 - c - czm) - z} dH(t)$$

$H(t)$: limiting eigenvalue distribution of Σ_N.

Local eigenvalue statistics

Most results assumed the covariance matrix Σ_N is a finite rank perturbation of the identity matrix.

This is one of the few results for non-spiked models.

Baik, Ben-Arous and Péché (0): eigenvalue correlation functions in terms of a determinantal formula.

Multiple Laguerre polynomials

Generalization of Laguerre polynomials that are orthogonal to multiple weights: $x^{M-N}e^{-Ma_j^{-1}x}$. E.g.

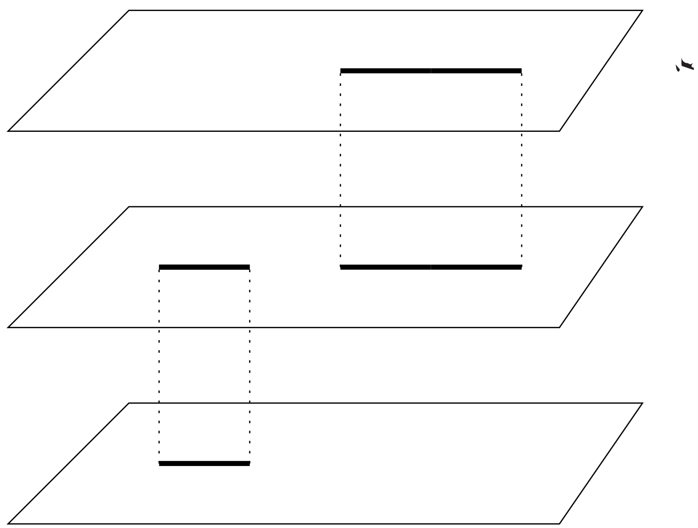
$$\int_0^\infty L_{n_1,n_2}(x)x^{i+M-N}e^{-Mx}dx = 0, \quad i = 0, \ldots, n_1 - 1$$

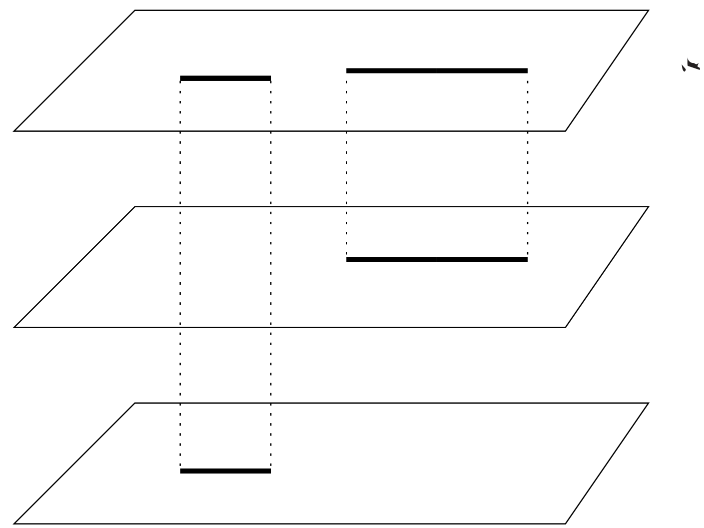
$$\int_0^\infty L_{n_1,n_2}(x)x^{i+M-N}e^{-Ma^{-1}x}dx = 0, \quad i = 0, \ldots, n_2 - 1$$

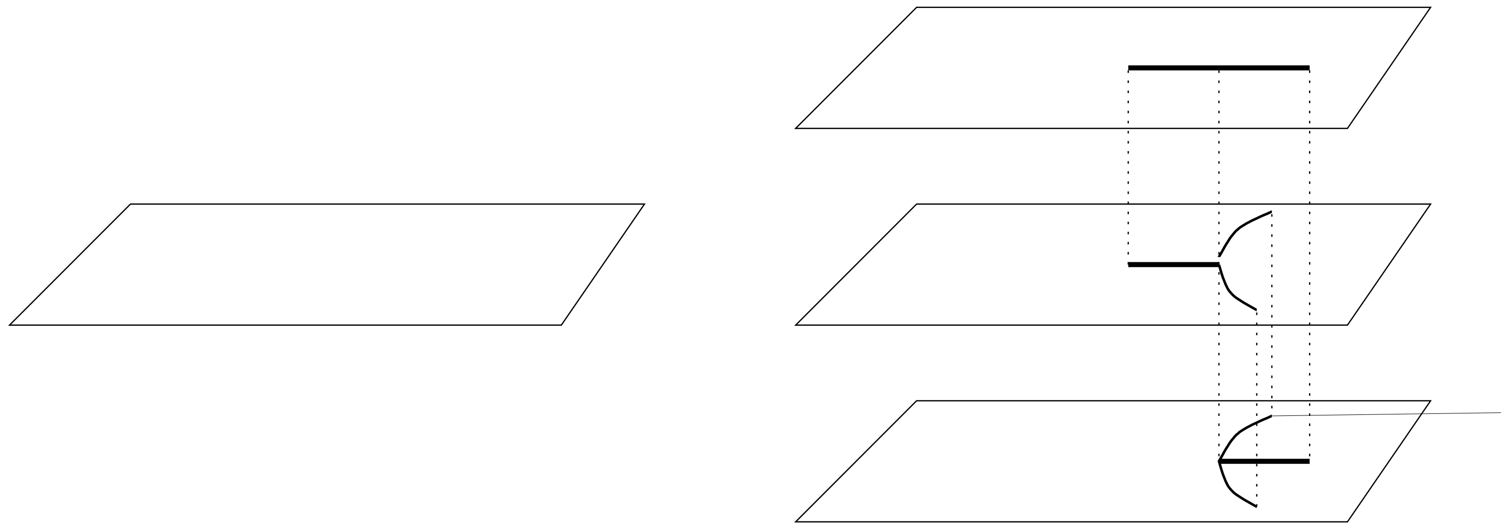
(Bleher and Kuijlaars, Desrosiers and Forrester) Correlation kernel of complex Wishart ensembles can be expressed in terms of multiple Laguerre polynomials.

Multiple Laguerre polynomials are solutions of Riemann-

Riemann surface depends on parameters and di cult to determine sheet structure in this case.







Stieltjes transform

De ne $\underline{F}$ by

$$\underline{F} = (\ - c)I_{0\ \ )} + cF$$

then the Stieltjes transform $m_{\underline{F}}$ satis es

$$m_{\underline{F}}(z) = -\left(z - c \ _{\mathbb{R}} \frac{tdH(t)}{\ + tm_{\underline{F}}}\right)$$

In our case, $dH(t)$ is

$$dH(x) = (\ - \ ) \ + \ \ _a$$

Gives us an algebraic equation

$$z = \ \frac{}{} - + c\frac{-}{\ +} + c\frac{a}{\ + a}$$

Observations:

. The equation has solutions behaves like

$$(z) \qquad -\frac{}{z} + O(z^{-}) \qquad z \to$$

$$(z) \qquad + \frac{c(\quad - \quad)}{z} + O(z^{-}) \qquad z \to$$

$$(z) \qquad -\frac{}{a} + \frac{c}{z} + O(z^{-}) \qquad z \to$$

. $m_{\underline{F}}$ is the solution . So has branch cut on the support of $\underline{F}$.

Lemma 1 If $z \in \mathrm{supp}(\underline{F})$, then $m \equiv m_{\underline{F}}(z)$ satisfies the following.

1. $m \quad \mathbb{R} \quad 0$ **;**

2. $-\frac{1}{m} \in \mathrm{supp}(H)$**;**

3. $z'(m) \quad 0$**.**

Conversely, if m satisfies 1-3, then $z \equiv z(m) \in \mathrm{supp}(\underline{F})$.

$z' \equiv 0$ are the potential end points of the support.

Zeros of the polynomials

$$a \quad (\quad - c) \quad + \quad (a \, (\quad - c \quad) + a(\quad - c(\quad - \quad))$$
$$+ \quad (\quad - c(\quad - \quad) + a \, (\quad - c \quad) + \quad a) \quad + \quad (\quad + a) \quad + \quad \quad 0$$

Discriminant positive, real roots , Discriminant negative, real roots .

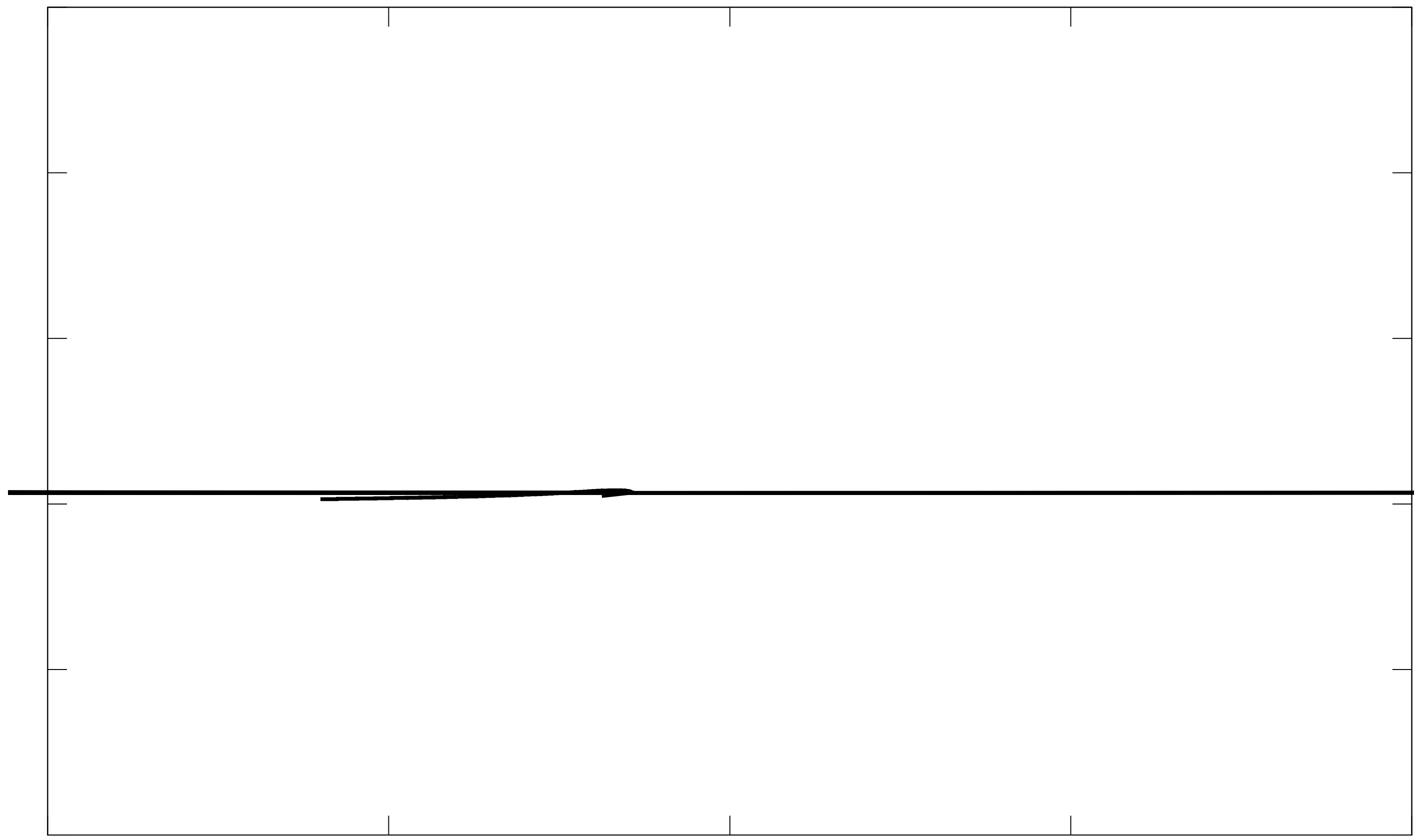
Complement of support: E.g. when there are real roots,

$$\text{supp}(\underline{F})^c \quad (- \quad 0) \quad (0 \quad) \quad (\quad) \quad (\quad)$$

$_k$ are given by

$$_k \quad -\frac{}{_k} + c\frac{-}{+ \quad _k} + c\frac{a}{+ a \quad _k}$$

So if we have , then support consists of intervals, otherwise support has interval.



Summary

Studied complex Wishart ensembles whose covariance matrix has N_0 eigenvalues and N eigenvalues a and $\frac{N}{M} \to c$, $\frac{N}{N} \to$.

One of the few cases when results was obtained for non-spiked models.

Correlation kernel given by Sine-kernel in bulk and Airy kernel in edge. Tracey-Widom distribution for largest eigenvalue.

Uses Stieltjes transform to overcome difficulties in the application of Riemann-Hilbert analysis.

When covariance matrix has nitely many distinct eigenvalues, Stieltjes transform gives a Riemann surface and the analysis can be generalized to the case when the Riemann surface has less than or equal to complex branch points.