Exact Minimum Eigenvalue Distribution of a Random Entangled Bipartite System

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Plan

Plan:

- A brief review of the physical system

  = Randomly coupled entangled bipartite quantum system

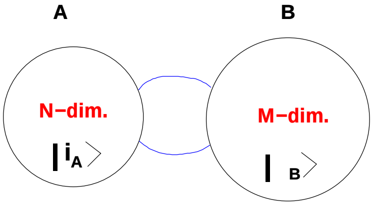
- Reduced density matrix and its eigenavalue statistics

- Minimum eigenvalue $\lambda_{\rm min} \to$ exact PDF

  proving, on the side, a recent conjecture by Znidaric (2007).

- Summary and Conclusions

Coupled Bipartite System



Coupled Bipartite System

$\mathbf{N \ll M}$

Composite System: $A \otimes B$

Any state: $$|\psi> = \sum_{i,\alpha} x_{i,\alpha} \, |i_A> \otimes |\alpha_B>$$

$X = [x_{i,\alpha}] \rightarrow (N \times M)$ rectangular Coupling matrix

Coupled Bipartite System

- $$|\psi> = \sum_{i,\alpha} x_{i,\alpha} \, |i_A> \otimes |\alpha_B>$$

- If $x_{i,\alpha} = a_i \, b_\alpha$ then

$$|\psi> = \sum_i a_i \, |i_A> \otimes \sum_\alpha b_\alpha \, |\alpha_B> = |\Phi_A> \otimes |\Phi_B>$$

$\longrightarrow$ Fully Un-entangled (factorised)

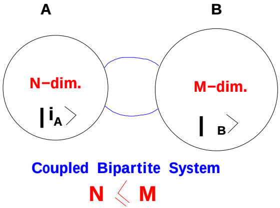
Otherwise $\longrightarrow$ Entangled (non-factorisable)

- Density matrix of the composite system

$$\boxed{\hat{\rho} = |\psi><\psi|} \quad \text{with } \mathrm{Tr}[\hat{\rho}] = 1$$

- Pure state: $\hat{\rho} = \sum_k p_k |\psi_k><\psi_k| \rightarrow$ not a Mixed state

Reduced Density Matrix of subsystem A:



- Reduced Density Matrix: $\hat{\rho}_A = \mathrm{Tr}_B[\hat{\rho}]$ with $\mathrm{Tr}[\hat{\rho}_A$

Summary:

- Schmidt decomposition: $$|\psi> = \sum_{i=1}^{N} \sqrt{\lambda_i}\, |\lambda_i^A> \otimes |\lambda_i^B>$$

- Reduced density matrix: $\hat{\rho}_A = \mathrm{Tr}_{\mathrm{B}}\left[|\psi><\psi|\right] = \sum_{i=1}^{N} \lambda_i\, |\lambda_i^A><\lambda_i^A|$

- $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \rightarrow$ eigenvalues of $W = XX^\dagger$ with

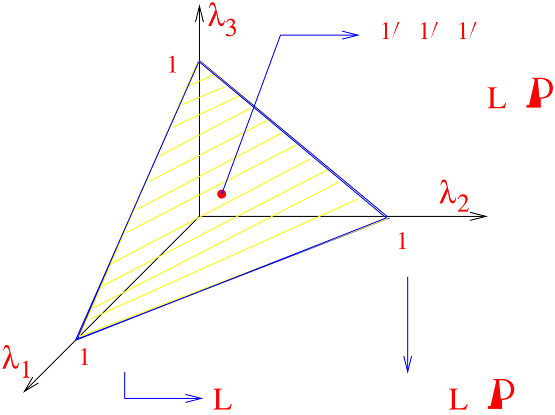
  $0 \leq \lambda_i \leq 1$ and $\sum_{i=1}^{N} \lambda_i = 1$

- Least entangled (unentangled): if $\lambda_1 = 1$ and $\lambda_2 = \lambda_3 = \ldots = \lambda_N = 0$

  $|\psi> = |\lambda_1^A> \otimes |\lambda_1^B>$ (fully factorised)

- Most entangled: if $\lambda_1 = \lambda_2 = \ldots = \lambda_N = 1/N$

A Simple Diagram for N=3



Minimum Eigenvalue $\lambda_{\min}$:

- Another important object: $\lambda_{\min}$ =

Randomly Coupled Bipartite System

$$|\psi> = \sum_{i,\alpha} x_{i,\alpha} \, |i_A> \otimes |\alpha_B> = \sum_{i=1}^{N} \sqrt{\lambda_i} \, |\lambda_i^A> \otimes |\lambda_i^B>$$

- $X = [x_{i,\alpha}]$ → entries are Random variables drawn from:

$$\text{Prob}[X] \propto \exp\left[-\frac{\beta}{2} \text{Tr}\left(X^\dagger X\right)\right]$$; $\beta = 2$ (complex) and $\beta = 1$ (real)

- $\{\lambda_i\}$ → random variables

- entropy $S = -\sum_{i=1}^{N} \lambda_i \ln(\lambda_i)$ → random variable

Znidaric Conjecture and our Results:

- For $M = N$ and $\beta = 2$, Znidaric (2007) conjectured:

$$\boxed{\lambda_{\min} = \frac{1}{N^3}} \text{ for all } N$$

- For $M = N$ and

Minimum Eigenvalue Distribution for the Real Case [redacted]

- For $M = N$ and $\beta = 1$ we p[redacted]

$$P_N(x) = A_N\, x^{-N/2}\,(1 - Nx)^{(N^2+N-4)/2}\;{}_2F_1\left(\frac{N+2}{2}, \frac{N-1}{2}, \frac{N^2+N-2}{2}, -\frac{1-Nx}{x}\right)$$

where

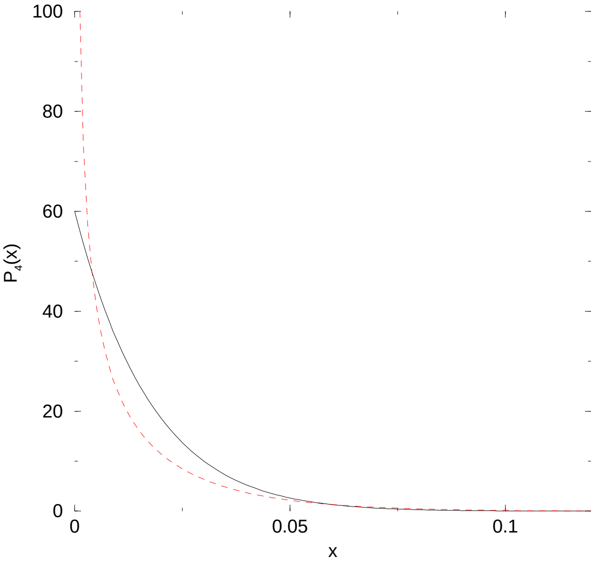
$$A_N = \frac{N\,\Gamma(N)\,\Gamma(N^2/2)}{2^{N-1}\,\Gamma(N/2)\,\Gamma((N^2+N-2)/2)}$$

- For example for $N = 2$: $P_2(x) = \frac{\sqrt{1-2x}}{x(1-x)}$ for $0 \le x \le 1/2$

- All moments calculated explicitly: For example the average:

$\mu_1(N) = x \quad 2$ (069.9626Tf6Tf7.86320Td[())1]TJ21179.9626Tf6.4169-2.878]T

Exact Minimum Eigenvalue PDF for N=4



Sketch of the Proof

- To compute the distribution of $\lambda_{\min} = \min(\lambda_1, \lambda_2, \ldots, \lambda_N)$ need
- the joint PDF $P(\lambda_1, \lambda_2, \ldots, \lambda_N)$
  where $\{\lambda_1, \lambda_2, \ldots, \lambda_N\} \to$ eigenvalues of the Wishart matrix $W = XX^\dagger$

  with an additional constraint $\boxed{\sum_{i=1}^{N} \lambda_i = 1}$

- Joint distribution of Wishart eigenvalues (James '64):

$$\boxed{P(\{\lambda_i\}) \propto \exp\left[-\frac{\beta}{2}\sum_{i=1}^{N}\lambda_i\right] \prod_i \lambda_i^{\frac{\beta}{2}(1+M-N)-1} \prod_{j<k} |\lambda_j - \lambda_k|^\beta}$$

$$\times\,\delta\left[\sum_{i=1}^{N}\lambda_i - 1\right]$$

(Llyod & Pagels '88, Zyczkowski & Sommers '2001)

- Cumulative distribution of $\lambda_{\min}$:
  $Q_N(x) = \mathrm{Prob}(\lambda_{\min} \geq x) = \mathrm{Prob}(\lambda_1 \geq x, \lambda_2 \geq x, \ldots \lambda_N \geq x)$

Cumulative distribution of λ_{min} for M = N

- For $M = N$ case

$$Q_N(x) = B_N \int_x^\infty \cdots \int_x^\infty \delta\left(\sum_{i=1}^N \lambda_i - 1\right) \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}-1} d\lambda_i$$

- Integral representation

$$\delta\left(\sum \lambda_i - 1\right) = \int \frac{ds}{2\pi i} e^{s(1-\sum \lambda_i)}$$

- $Q_N(x) = B_N \int \frac{ds}{2\pi i} e^s I_N(x,s)$ where

$$I_N(s,x) = \int_x^\infty \cdots \int_x^\infty e^{-s\sum \lambda_i} \prod_{j<k} |\lambda_j - \lambda_k|^\beta \prod_{i=1}^N \lambda_i^{\frac{\beta}{2}-1} d\lambda_i$$

$M = N$ and $\beta = 2$ Case

- For $\beta = 2$ the integral simplifies:

$$I_N(s,x) = \int_x^\infty \dots \int_x^\infty e^{-s\sum \lambda_i} \prod_{j<k} (\lambda_j - \lambda_k)^2 \prod_{i=1}^{N} d\lambda_i$$

- Using a shift: $z_i = s(\lambda_i - x)$ → Selberg Integral

- $$I_N(s,x) = \frac{e^{-sNx}}{s^{N^2}} \prod_{j=0}^{N-1} \Gamma(j+2)\Gamma(j+1)$$

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Summary and Conclusions:

- Exact PDF of the minimum eigenvalue λ_{min} of the reduced density matrix of a randomly coupled bipartite system of equal sizes for $\beta = 1$ and $\beta = 2$

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